Algebraic treatment of second Poschl-Teller, Morse-Rosen and Eckart equations

1987 J. Phys. A: Math. Gen. 20 4083

(http://iopscience.iop.org/0305-4470/20/13/017)

J. Phys. A: Math. Gen. 20 (1987) 4083-4096. Printed in the UK

Algebraic treatment of second Pöschl–Teller, Morse–Rosen and Eckart equations

A O Barut†‡, Akira Inomata§ and Raj Wilson‖¶

† International Centre for Theoretical Physics, Miramare, Trieste, Italy
§ Department of Physics, State University of New York, Albany, NY 12222, USA
‖ Department of Physics, Università di Trento, 38150 Povo di Trento, Italy

Received 24 February 1987

Abstract. The method of an earlier paper is applied to the non-compact case to solve a family of second Pöschl-Teller, Morse-Rosen and Eckart equations with quantised coupling constants. Both discrete and continuous spectra, bound state and scattering wavefunctions (transmission coefficients) are found from the matrix elements of group representations.

1. Introduction

In the preceding paper we proposed (Barut *et al* 1987a, hereafter referred to as I) a new method of algebraisation of physical equations. In this method the given equation is expressed in terms of invariant bilinear forms of ladder operators of certain Lie group matrix elements. These ladder operators, which may be constructed from Infeld-Hull-Miller factorisations, close under a Lie algebra. From the enveloping algebra we obtain the required energy spectrum, and the unitary representations of the corresponding Lie group give the exact normalised solutions of the equation. We used this method to study the first Pöschl-Teller equation for diatomic molecules and in this paper we discuss the second Pöschl-Teller equation, the Morse-Rosen equation for polyatomic molecules and the Eckart equation for the electron penetration barrier. Algebraisation of these three equations involves the algebra of the non-compact Lie group SO(2, 1) and its unitary representations (Bargmann 1947, Kunze and Stein 1960, Barut and Fronsdal 1965, Toller 1965, Barut and Phillips 1968, Mukunda 1969, Lindblad and Nagel 1970, Rühl 1970, Biedenharn and Louck 1981). In each case we obtain the correct energy spectrum and the eigensolutions. We also point out that the action of the ladder operators we construct is similar to the action of the covariant differentiation operator ('edth' operator) of the Bondi-Metzner-Sachs group (Newman and Penrose 1966, Goldberg *et al* 1967). For the second Pöschl-Teller equation only the special case $\kappa = 0$ was solved before. We treat both the bound states and scattering solutions.

2. The second Pöschl–Teller equation

The second Pöschl-Teller equation is

$$\left[\frac{\partial^2}{\partial r^2} - \alpha_1^2\left(\frac{\kappa(\kappa-1)}{\sinh^2 \alpha_1 r} - \frac{\lambda(\lambda+1)}{\cosh^2 \alpha_1 r}\right) + \frac{2ME}{\hbar^2}\right]\psi(r) = 0 \qquad r \in [0, \infty). \tag{2.1}$$

‡ Permanent address: Department of Physics, University of Colorado, Boulder, CO 80309, USA.
¶ Present address: Department of Mathematics, University of Texas, San Antonio, TX 78285, USA.

We may assume $\lambda > \kappa$, for if $\lambda < \kappa$, we can change $\lambda \to -\lambda - 1$, because the equation remains unchanged under $\kappa \to -\kappa + 1$ and $\lambda \to -\lambda - 1$. Furthermore the mappings $\{r \to ir, \lambda \to \lambda - 1, E \to -E\}$ or $\{\alpha_1 \to i\alpha_1, \lambda \to \lambda - 1\}$ provide the *analytical continuation* to the first Pöschl–Teller equation. We introduce a change of parameters: $\kappa = -m - g + \frac{1}{2}$, $\lambda = m - g - \frac{1}{2}$, $\beta = 2\alpha_1 r$ and (2.1) becomes

$$\left[\frac{\partial^2}{\partial\beta^2} - \frac{1}{4}\left(\frac{(m+g+\frac{1}{2})(m+g-\frac{1}{2})}{\sinh^2\beta/2} - \frac{(m-g-\frac{1}{2})(m-g+\frac{1}{2})}{\cosh^2\beta/2}\right) + \Lambda\right]\psi(r) = 0$$

$$\Lambda \equiv ME/2\hbar^2\alpha_1^2. \tag{2.2}$$

Following Infeld–Hull–Miller factorisation of type A, we define operators M^+, M^-, M_3 acting in the same space of functions to be determined as

$$\begin{aligned} M^+\psi_{m,g} &= \exp(i\alpha)\left(-\frac{\partial}{\partial\beta} + \tfrac{1}{2}(m+g+\tfrac{1}{2})\coth\frac{\beta}{2} + \tfrac{1}{2}(m-g+\tfrac{1}{2})\tanh\frac{\beta}{2}\right)\psi_{m,g} \\ &= [\Lambda + (m+\tfrac{1}{2})^2]^{1/2}\psi_{m+1,g} \\ M^-\psi_{m,g} &= \exp(-i\alpha)\left(\frac{\partial}{\partial\beta} + \tfrac{1}{2}(m+g-\tfrac{1}{2})\coth\frac{\beta}{2} + \tfrac{1}{2}(m-g-\tfrac{1}{2})\tanh\frac{\beta}{2}\right)\psi_{m,g} \\ &= [\Lambda + (m-\tfrac{1}{2})^2]^{1/2}\psi_{m-1,g} \\ M_3\psi_{m,g} &= -i\frac{\partial}{\partial\alpha}\psi_{m,g} = m\psi_{m,g} \qquad \alpha \in [0, 2\pi). \end{aligned} \tag{2.3}$$

The operators M^+, M^-, M_3 close under SU(1, 1) algebra (Bargmann 1947, Barut and Fronsdal 1965) satisfying the relations:

$$[M^+, M^-] = -2M_3$$
$$[M^\pm, M_3] = \mp M^\pm.$$

The Casimir product is

$$Q_{SU(1,1)}\psi_{m,g} = (-M^+M^- + M_3M_3 - M_3)\psi_{m,g} = (-\Lambda - \tfrac{1}{4})\psi_{m,g} \equiv l(l-1)\psi_{m,g}.$$

We have denoted the eigenvalue of Q by $l(l-1)$, hence we obtain

$$\Lambda = -(l - \tfrac{1}{2})^2 \qquad l = m - n \qquad m > n \in \mathbb{N}$$

which gives immediately the energy spectrum

$$E_n = -\frac{2\alpha_1^2\hbar^2}{M}(l-\tfrac{1}{2})^2 = -\frac{\alpha_1^2\hbar^2}{2M}(\lambda - \kappa - 2n)^2 \qquad n = 0, 1, \ldots, <(\lambda-\kappa)/2. \tag{2.4}$$

The discrete energy spectrum is obtained from the discrete representations

$$D_l^+ \qquad l = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \qquad (m, g) = l, l+1, \ldots$$
$$D_l^- \qquad l = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \qquad (m, g) = -l, -l-1, \ldots.$$

Equation (2.2) can now be written in the algebraic form

$$[Q_{SU(1,1)} - l(l-1)]\psi_{m,g} = 0. \tag{2.5}$$

We now define

$$\begin{aligned} -i\frac{\partial}{\partial\gamma}\psi_{m,g} &= g\psi_{m,g} \\ M^\pm &= M_1 \pm iM_2 \end{aligned} \tag{2.6}$$

and obtain the SO(2, 1) generators

$$\begin{aligned}
M_1 &= -\mathrm{i}\cos\alpha\coth\beta\frac{\partial}{\partial\alpha} - \mathrm{i}\sin\alpha\frac{\partial}{\partial\beta} - \mathrm{i}\frac{\cos\alpha}{\sinh\beta}\frac{\partial}{\partial\gamma} + \tfrac{1}{2}\mathrm{i}\sin\alpha\coth\beta \\
M_2 &= -\mathrm{i}\sin\alpha\coth\beta\frac{\partial}{\partial\alpha} + \mathrm{i}\cos\alpha\frac{\partial}{\partial\beta} - \mathrm{i}\frac{\sin\alpha}{\sinh\beta}\frac{\partial}{\partial\gamma} - \frac{\mathrm{i}}{2}\cos\alpha\coth\beta \\
M_3 &= -\mathrm{i}\frac{\partial}{\partial\alpha} \\
[M_1, M_2] &= -\mathrm{i}M_3 \qquad [M_3, M_1] = \mathrm{i}M_2 \qquad [M_2, M_3] = \mathrm{i}M_1 .
\end{aligned} \tag{2.7}$$

As in I we shall now introduce the second set of ladder operators. We go back to (2.2) which can also be obtained from (2.1) by interchanging m and g such that $\kappa = -m - g + \frac{1}{2}$, $\lambda = -m + g - \frac{1}{2}$. Again, factorisation of type A leads to the ladder operators

$$\begin{aligned}
G^+\psi_{m,g} &= -\exp(\mathrm{i}\gamma)\left(-\frac{\partial}{\partial\beta} + \tfrac{1}{2}(g+m+\tfrac{1}{2})\coth\frac{\beta}{2} + \tfrac{1}{2}(g-m+\tfrac{1}{2})\tanh\frac{\beta}{2}\right)\psi_{m,g} \\
&= [\Lambda + (g+\tfrac{1}{2})^2]^{1/2}\psi_{m,g+1} \\
G^-\psi_{m,g} &= -\exp(-\mathrm{i}\gamma)\left(\frac{\partial}{\partial\beta} + \tfrac{1}{2}(g+m-\tfrac{1}{2})\coth\frac{\beta}{2} + \tfrac{1}{2}(g-m-\tfrac{1}{2})\tanh\frac{\beta}{2}\right)\psi_{m,g} \\
&= [\Lambda + (g-\tfrac{1}{2})^2]^{1/2}\psi_{m,g-1} \\
G^3\psi_{m,g} &= -\mathrm{i}\frac{\partial}{\partial\gamma}\psi_{m,g} = g\psi_{m,g} \qquad \gamma\in[0, 2\pi).
\end{aligned} \tag{2.8}$$

The operators G^+, G^-, G_3 form another SU(1, 1) algebra and the Casimir product leads to the same energy spectrum (2.4) and the same algebraic form (2.5).

We define $G^{\pm} = G_1 \mp \mathrm{i}G_2$ and obtain the generators G_i, $i = 1, 2, 3$, as

$$\begin{aligned}
G_1 &= \mathrm{i}\frac{\cos\gamma}{\sinh\beta}\frac{\partial}{\partial\alpha} + \mathrm{i}\sin\gamma\frac{\partial}{\partial\beta} + \mathrm{i}\cos\gamma\coth\beta\frac{\partial}{\partial\gamma} - \frac{\mathrm{i}}{2}\sin\gamma\coth\beta \\
G_2 &= -\mathrm{i}\frac{\sin\gamma}{\sinh\beta}\frac{\partial}{\partial\alpha} + \mathrm{i}\cos\gamma\frac{\partial}{\partial\beta} - \mathrm{i}\sin\gamma\coth\beta\frac{\partial}{\partial\gamma} - \frac{\mathrm{i}}{2}\cos\gamma\coth\beta \\
G_3 &= -\mathrm{i}\frac{\partial}{\partial\gamma} \\
[G_1, G_2] &= \mathrm{i}G_3 \qquad [G_3, G_1] = -\mathrm{i}G_2 \qquad [G_2, G_3] = -\mathrm{i}G_1 .
\end{aligned} \tag{2.9}$$

As expected the above commutation relations differ from those of (2.7) by an extra negative sign. From (2.7) and (2.9) we conclude that

$$\begin{aligned}
&[M_i, G_j] = 0 \qquad i, j = 1, 2, 3 \\
&G_i = \sum_j H_{ji}(\alpha, \beta, \gamma)M_j \\
&H(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\cosh\beta & 0 & \sinh\beta \\ 0 & 1 & 0 \\ \sinh\beta & 0 & -\cosh\beta \end{pmatrix}\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{2.10}$$

Thus the solutions of (2.1) are the eigenfunctions satisfying

$$\begin{aligned} &Q_{SU(1,1)}\psi(r)=l(l-1)\psi(r) \qquad \psi=\exp(im\alpha)\exp(ig\gamma)\psi_{m,g} \\ &M_3\psi_{m,g}=m\psi_{m,g} \\ &G_3\psi_{m,g}=g\psi_{m,g}. \end{aligned} \tag{2.11}$$

From (2.3) and (2.8) we obtain the following recurrence relations:

$$\begin{aligned} \left(\frac{g+m\cosh\beta}{\sinh\beta}\right)\psi_{m,g}(\beta)&=\tfrac{1}{2}[(l+m)(-l+m+1)]^{1/2}\psi_{m+1,g}(\beta) \\ &\quad+\tfrac{1}{2}[(l+m-1)(-l+m)]^{1/2}\psi_{m-1,g}(\beta) \\ \left(\frac{m+g\cosh\beta}{\sinh\beta}\right)\psi_{m,g}(\beta)&=-\tfrac{1}{2}[(l+g)(-l+g+1)]^{1/2}\psi_{m,g+1}(\beta) \\ &\quad-\tfrac{1}{2}[(l+g-1)(-l+g)]^{1/2}\psi_{m,g-1}(\beta) \\ \left(\frac{\partial}{\partial\beta}-\tfrac{1}{2}\coth\beta\right)\psi_{m,g}(\beta)&=-\tfrac{1}{2}[(l+m)(-l+m+1)]^{1/2}\psi_{m+1,g}(\beta) \\ &\quad+\tfrac{1}{2}[(l+m-1)(-l+m)]^{1/2}\psi_{m-1,g}(\beta) \\ &=\tfrac{1}{2}[(l+g)(-l+g+1)]^{1/2}\psi_{m,g+1}(\beta) \\ &\quad-\tfrac{1}{2}[(l+g-1)(-l+g)]^{1/2}\psi_{m,g-1}(\beta). \end{aligned} \tag{2.12}$$

In this case, in contrast to I, because we have a non-compact group, the crucial observation again is to compare (2.12) with the recurrence relations for the Bargmann function given by (Schneider and Wilson 1979)

$$\begin{aligned} \left(\frac{n-n'\cosh\theta}{\sinh\theta}\right)V^l_{n',n}(\theta)&=\tfrac{1}{2}[(l+n')(-l+n'+1)]^{1/2}V^l_{n'+1,n}(\theta) \\ &\quad+\tfrac{1}{2}[(l+n'-1)(-l+n')]^{1/2}V^l_{n'-1,n}(\theta) \\ \left(\frac{n'-n\cosh\theta}{\sinh\theta}\right)V^l_{n',n}(\theta)&=-\tfrac{1}{2}[(l+n)(-l+n+1)]^{1/2}V^l_{n',n+1}(\theta) \\ &\quad-\tfrac{1}{2}[(l+n-1)(-l+n)]^{1/2}V^l_{n',n-1}(\theta) \\ \frac{\partial}{\partial\theta}V^l_{n',n}(\theta)&=\tfrac{1}{2}[(l+n')(-l+n'+1)]^{1/2}V^l_{n'+1,n}(\theta) \\ &\quad-\tfrac{1}{2}[(l+n'-1)(-l+n')]^{1/2}V^l_{n'-1,n}(\theta) \\ &=-\tfrac{1}{2}[(l+n)(-l+n+1)]^{1/2}V^l_{n'n+1}(\theta) \\ &\quad+\tfrac{1}{2}[(l+n-1)(-l+n)]^{1/2}V^l_{n'n-1}(\theta). \end{aligned} \tag{2.13}$$

The comparison of (2.12) and (2.13) gives the exact normalised solution

$$\begin{aligned} &\psi_{m,g}(\beta)=(2l-1)^{1/2}\sinh^{1/2}\beta\, V^l_{m,g}(-i\pi-\beta) \\ &\int_0^\infty \psi^*_{m',g'}(\beta)\psi_{m,g}(\beta)\,d\beta=\delta_{m',m}\delta_{g',g} \\ &\int_0^\infty V^{l*}_{n',m'}(\theta)V^l_{n,m}(\theta)\sinh\theta\,d\theta=(2l-1)^{-1/2}\delta_{n'n}\delta_{m',m} \qquad l>\tfrac{1}{2}. \end{aligned} \tag{2.14}$$

In order to express the solution in an explicit form we use the following (Barut and Wilson 1976):

$$\begin{gathered} V^l_{m,n}(\theta)=(-1)^{n-m}V^l_{n,m}(\theta)=(-1)^{n-m}V^l_{m,n}(\theta) \\ V^l_{m,n}(\theta+\mathrm{i}\pi)=(-1)^{m-l+1/2}V^l_{m,-n}(\theta) \\ V^l_{m,n}(\theta)=\left[\binom{m-l}{n-l}\binom{m+l-1}{n+l-1}\right]^{1/2}(\tanh\tfrac{1}{2}\theta)^{m-n}(\cosh\tfrac{1}{2}\theta)^{-2n} \\ \times{}_2F_1[l-n,1-n-l;1+m-n;-\sinh^2\tfrac{1}{2}\theta] \end{gathered} \tag{2.15}$$

and in terms of the original variables r, κ, λ ($\beta=2\alpha_1 r$, $m=\frac{1}{2}(\lambda-\kappa+1)$, $g=-\frac{1}{2}(\kappa+\lambda)$, $l=\frac{1}{2}(\lambda-\kappa+1-2n)$) we obtain, after using Euler's identity for ${}_2F_1$ functions, the final form

$$\begin{aligned} \psi(r)=\Bigg[&4\alpha_1(2l-1)\binom{\frac{1}{2}(\lambda-\kappa+1)-l}{\frac{1}{2}(\lambda+\kappa)-l} \\ &\times\binom{\frac{1}{2}(\lambda-k+1)+l-1}{\frac{1}{2}(\lambda+\kappa)+l-1}\Bigg]^{1/2}(\sinh\alpha_1 r)^{1-\kappa}(\cosh\alpha_1 r)^{\lambda+1} \\ &\times{}_2F_1\left[\left(\frac{\lambda-\kappa}{2}-l+1\right)+\tfrac{1}{2},\left(\frac{\lambda-\kappa}{2}+l\right)+\tfrac{1}{2};\tfrac{3}{2}-\kappa;-\sinh^2\alpha_1 r\right]. \end{aligned} \tag{2.16}$$

This is one of the standard solutions and the related second standard solution is obtained by replacing the r-dependent part by

$$(\sinh\alpha_1 r)^{\kappa}(\cosh\alpha_1 r)^{\lambda+1}{}_2F_1\left[\left(\frac{\lambda+\kappa}{2}-l+1\right),\left(\frac{\lambda+\kappa}{2}+l\right);\tfrac{1}{2}+\kappa;-\sinh^2\alpha_1 r\right]. \tag{2.17}$$

Thus we have obtained the exact solution to (2.1) for a finite number of bound states satisfying the discrete spectrum (2.4). For the special case $\kappa=0$ our result is in agreement with the earlier result using non-algebraic methods (Flügge 1971). However the latter gives only the r-dependent part of the solution without the appropriate coefficients.

Historically the equations similar to (2.1) were studied by Weyl (1909). According to Weyl's criterion the equation has eigensolutions if $\alpha\geqslant\gamma\geqslant 1$ where $\alpha=1+|-\kappa+\frac{1}{2}|+|\lambda+\frac{1}{2}|$, $\gamma=1+|-\kappa+\frac{1}{2}|$ and it has a discrete eigenspectrum as given by (2.4) if $\gamma-\alpha/2<0\Rightarrow 1-|-\kappa+\frac{1}{2}|-|\lambda+\frac{1}{2}|<0$ and the eigenfunctions are square integrable as in (2.14). Furthermore (2.1) has also a continuous spectrum if $\gamma-\alpha/2\geqslant 0\Rightarrow 1+|-\kappa+\frac{1}{2}|-|\lambda+\frac{1}{2}|\geqslant 0$. This can be immediately obtained by using the continuous principal representation of SU(1, 1): C^0_l: $l=\frac{1}{2}+\mathrm{i}t$, $t\in(0,\infty)$ or $l=\frac{1}{2}-\mathrm{i}t$, $t\in(-\infty,0)$; $(m,g)=0,\pm1,\pm2,\ldots$ and $C^{1/2}_l$ with $(m,g)=\pm\frac{1}{2},\pm\frac{3}{2},\ldots$. The C^0_l and $C^{1/2}_l$ representations of Bargmann (1947) respectively correspond to $U^+(g,\frac{1}{2}+\mathrm{i}t)$ and $U^-(g,\frac{1}{2}+\mathrm{i}t)$ representations of Kunze and Stein (1960). From (2.14) we obtain the continuous energy spectrum

$$E_s=\frac{2\alpha_1^2\hbar^2}{M}t^2 \qquad t\in(-\infty,\infty). \tag{2.18}$$

The continuum solutions which are the scattering states as obtained from (2.16) are not square integrable. However, it is known (Bargmann 1947, Kunze and Stein 1960, Barut and Phillips 1968, Mukunda 1969, Lindblad and Nagel 1970, Rühl 1970) that for a fixed pair (m,g) or (κ,λ) the linear combinations of solutions of the type (2.16) and (2.17) are dense in the Hilbert space of square integral functions over the half-line $r\in[0,\infty]$.

Thus from (2.16) and (2.17) and using Kummer's identity for ${}_2F_1$ functions we obtain ($\sigma = \mathrm{i}t$)

$$\psi(r) = \hat{A}(\sinh \alpha_1 r)^{\kappa}(\cosh \alpha_1 r)^{\lambda+1}$$
$$\times {}_2F_1[(\tfrac{1}{2}+\tfrac{1}{2}(\kappa+\lambda)+\sigma), (\tfrac{1}{2}+\tfrac{1}{2}(\kappa+\lambda)-\sigma); (\tfrac{1}{2}+\kappa); -\sinh^2 \alpha_1 r]$$
$$+ \hat{B}(\sinh \alpha_1 r)^{1-\kappa}(\cosh \alpha_1 r)^{\lambda+1}$$
$$\times {}_2F_1[(\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)+\sigma), (\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)-\sigma); (\tfrac{3}{2}-\kappa); -\sinh^2 \alpha_1 r]$$

where

$$\hat{A} = \left(\frac{2|\Gamma(\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)+\sigma]|^2}{|\Gamma[\tfrac{1}{2}+\tfrac{1}{2}(\kappa+\lambda)+\sigma]|^2}\right)^{1/2}$$
$$\times \frac{\exp[\mathrm{i}\pi(\tfrac{1}{2}-\kappa)]\Gamma(\tfrac{1}{2}-\kappa)\exp(\mathrm{i}\pi 2\sigma)\Gamma(1-2\sigma)\Gamma(2\sigma)}{|\Gamma[\tfrac{1}{2}-\tfrac{1}{2}(\kappa+\lambda)+\sigma]|^2|\Gamma[\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)+\sigma]|^2}$$
$$\hat{B} = \left(\frac{2|\Gamma[\tfrac{1}{2}+\tfrac{1}{2}(\kappa+\lambda)+\sigma]|^2}{|\Gamma[\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)+\sigma]|^2}\right)^{1/2} \tag{2.19}$$
$$\times \frac{\exp[\mathrm{i}\pi(\kappa-\tfrac{1}{2})]\Gamma(\kappa-\tfrac{1}{2})\exp(-\mathrm{i}\pi 2\sigma)\Gamma(1+2\sigma)\Gamma(-2\sigma)}{|\Gamma[\tfrac{1}{2}+\tfrac{1}{2}(\kappa+\lambda)+\sigma]|^2|\Gamma[\tfrac{1}{2}-\tfrac{1}{2}(\lambda-\kappa+1)+\sigma]|^2}.$$

This solution can be expressed in another form as obtained by Bargmann (1947) as follows:

$$\psi(r) = \check{A}(\tanh \alpha_1 r)^{\lambda}(\sinh \alpha_1 r)^{2\sigma}$$
$$\times {}_2F_1[(\tfrac{1}{2}-\tfrac{1}{2}(\lambda-\kappa+1)-\sigma), (\tfrac{1}{2}-\tfrac{1}{2}(\kappa+\lambda)-\sigma); (1-2\sigma); -\sinh^{-2} \alpha_1 r]$$
$$+ \check{B}(\tanh \alpha_1 r)^{\lambda}(\sinh \alpha_1 r)^{-2\sigma}$$
$$\times {}_2F_1[(\tfrac{1}{2}-\tfrac{1}{2}(\lambda-\kappa+1)+\sigma), (\tfrac{1}{2}-\tfrac{1}{2}(\kappa+\lambda)+\sigma); (1+2\sigma); -\sinh^{-2} \alpha_1 r]$$

where

$$\check{A} = \left(\frac{2|\Gamma[\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)+\sigma]|^2}{|\Gamma[\tfrac{1}{2}+\tfrac{1}{2}(\kappa+\lambda)+\sigma]|^2}\right)^{1/2} \frac{\Gamma(2\sigma)\exp(\mathrm{i}\pi 2\sigma)}{\Gamma[\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)+\sigma]\Gamma[\tfrac{1}{2}-\tfrac{1}{2}(\kappa+\lambda)+\sigma]}$$
$$\check{B} = \left(\frac{2|\Gamma[\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)+\sigma]|^2}{|\Gamma[\tfrac{1}{2}+\tfrac{1}{2}\kappa+\lambda+\sigma]|^2}\right)^{1/2} \frac{\Gamma(-2\sigma)\exp(-\mathrm{i}\pi 2\sigma)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)-\sigma)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}(\kappa+\lambda)-\sigma)}. \tag{2.20}$$

Solutions obtained in (2.19) and (2.20) are not square integrable and they are essentially obtained from suitable linear combinations of $(\sinh\beta)^{1/2}\ V^{l}_{mg}(-\mathrm{i}\pi-\beta)$ with $l=\tfrac{1}{2}+\sigma$ and $l=\tfrac{1}{2}-\sigma$, $\sigma=\mathrm{i}t$. Since the Bargmann functions are asymptotically exponential it is possible to obtain square integrable functions by forming asymptotic 'wavepackets'. To obtain this conveniently we express (2.19) or (2.20) as follows:

$$\psi(r) = \tfrac{1}{2}A(\tanh \alpha_1 r)^{\kappa}(\cosh \alpha_1 r)^{2\sigma}$$
$$\times {}_2F_1[(\tfrac{1}{2}+\tfrac{1}{2}(\kappa+\lambda)-\sigma), (\tfrac{1}{2}-\tfrac{1}{2}(\lambda-\kappa+1)-\sigma); (\tfrac{1}{2}+\kappa); \tanh^2 \alpha_1 r]$$
$$+\tfrac{1}{2}A(\tanh \alpha_1 r)^{\kappa}(\cosh \alpha_1 r)^{-2\sigma}$$
$$\times {}_2F_1[(\tfrac{1}{2}+\tfrac{1}{2}(\kappa+\lambda)+\sigma), (\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)+\sigma); (\tfrac{1}{2}+\kappa); \tanh^2 \alpha_1 r]$$
$$+\tfrac{1}{2}B(\tanh \alpha_1 r)^{1-\kappa}(\cosh \alpha_1 r)^{2\sigma}$$
$$\times {}_2F_1[(\tfrac{1}{2}-\tfrac{1}{2}(\kappa+\lambda)-\sigma), (\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)-\sigma); (\tfrac{3}{2}-\kappa); \tanh^2 \alpha_1 r]$$
$$+\tfrac{1}{2}B(\tanh \alpha_1 r)^{1-\kappa}(\cosh \alpha_1 r)^{2\sigma}$$
$$\times {}_2F_1[(\tfrac{1}{2}-\tfrac{1}{2}(\kappa+\lambda)+\sigma), (\tfrac{1}{2}+\tfrac{1}{2}(\lambda-\kappa+1)+\sigma); (\tfrac{3}{2}-\kappa); \tanh^2 \alpha_1 r] \tag{2.21}$$

where A and B are some arbitrary constants depending on κ, λ, σ and to be fixed by physical requirements or by asymptotic square integrability of $\psi(r)$. In fact several M functions as in (2.19) or (2.20) are hidden inside A and B; they will, after all, become irrelevant at the end. Since as $r \to \pm\infty$, $\tanh \alpha_1 r \to \pm 1$ we can simplify the ${}_2F_1$ functions in (2.21) by using Gauss's theorem:

$$ {}_2F_1[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \mathrm{Re}(c-a-b) > 0. $$

We obtain from (2.21) (from (2.18) $\sigma = \mathrm{i}\hat{k}/2\alpha$, $\hat{k} = (2ME)^{1/2}/h$)

$$ \lim_{r\to\infty} \psi(r) = \psi_+(r) = A_+ \exp(\mathrm{i}\hat{k}r) + B_+ \exp(-\mathrm{i}\hat{k}r) $$

$$ \lim_{r\to-\infty} \psi(r) = \psi_-(r) = A_- \exp(\mathrm{i}\hat{k}r) + B_- \exp(-\mathrm{i}\hat{k}r) $$

where

$$ \begin{aligned} A_+ &= \frac{A}{2} \exp[-(\mathrm{i}\hat{k}/\alpha_1)\ln 2] \frac{\Gamma(\mathrm{i}\hat{k}/\alpha_1)\Gamma(\frac{1}{2}+\kappa)}{\Gamma[\frac{1}{2}+\frac{1}{2}(\kappa+\lambda)+\mathrm{i}\hat{k}/2\alpha_1]\Gamma[\frac{1}{2}-\frac{1}{2}(\lambda-\kappa+1)+\mathrm{i}\hat{k}/2\alpha_1]} \\ &\quad + \frac{B}{2} \exp[-(\mathrm{i}\hat{k}/\alpha_1)\ln 2] \frac{\Gamma(\mathrm{i}\hat{k}/\alpha_1)\Gamma(\frac{3}{2}-\kappa)}{\Gamma[\frac{1}{2}-\frac{1}{2}(\kappa+\lambda)+\mathrm{i}\hat{k}/2\alpha_1]\Gamma[\frac{1}{2}+\frac{1}{2}(\lambda-\kappa+1)+\mathrm{i}\hat{k}/2\alpha_1]} \\ B_+ &= \frac{A}{2} \exp[(\mathrm{i}\hat{k}/\alpha_1)\ln 2] \frac{\Gamma(-\mathrm{i}\hat{k}/\alpha_1)\Gamma(\frac{1}{2}+\kappa)}{\Gamma[\frac{1}{2}+\frac{1}{2}(\kappa+\lambda)-\mathrm{i}\hat{k}/2\alpha_1]\Gamma[\frac{1}{2}-\frac{1}{2}(\lambda-\kappa+1)-\mathrm{i}\hat{k}/2\alpha_1]} \\ &\quad + \frac{B}{2} \exp[(\mathrm{i}\hat{k}/\alpha_1)\ln 2] \frac{\Gamma(-\mathrm{i}\hat{k}/\alpha_1)\Gamma(\frac{3}{2}-\kappa)}{\Gamma[\frac{1}{2}-\frac{1}{2}(\kappa+\lambda)-\mathrm{i}\hat{k}/2\alpha_1]\Gamma[\frac{1}{2}+\frac{1}{2}(\lambda-\kappa+1)-\mathrm{i}\hat{k}/2\alpha_1]} \\ A_- &= \frac{A}{2} \exp[(\mathrm{i}\hat{k}/\alpha_1)\ln 2] \frac{\Gamma(-\mathrm{i}\hat{k}/\alpha_1)\Gamma(\frac{1}{2}+\kappa)(-1)^{\kappa}}{\Gamma[\frac{1}{2}+\frac{1}{2}(\kappa+\lambda)-\mathrm{i}\hat{k}/2\alpha_1]\Gamma[\frac{1}{2}-\frac{1}{2}(\lambda-\kappa+1)-\mathrm{i}\hat{k}/2\alpha_1]} \\ &\quad - \frac{B}{2} \exp[(\mathrm{i}\hat{k}/2\alpha_1)\ln 2] \frac{\Gamma(-\mathrm{i}\hat{k}/\alpha_1)\Gamma(\frac{3}{2}-\kappa)(-1)^{-\kappa}}{\Gamma[\frac{1}{2}-\frac{1}{2}(\kappa+\lambda)-\mathrm{i}\hat{k}/2\alpha_1]\Gamma[\frac{1}{2}+\frac{1}{2}(\lambda-\kappa+1)-\mathrm{i}\hat{k}/2\alpha_1]} \\ B_- &= \frac{A}{2} \exp[-(\mathrm{i}\hat{k}/\alpha_1)\ln 2] \frac{\Gamma(\mathrm{i}\hat{k}/\alpha_1)\Gamma(\frac{1}{2}+\kappa)(-1)^{\kappa}}{\Gamma[\frac{1}{2}+\frac{1}{2}(\kappa+\lambda)+\mathrm{i}\hat{k}/2\alpha_1]\Gamma[\frac{1}{2}-\frac{1}{2}(\lambda-\kappa+1)+\mathrm{i}\hat{k}/2\alpha_1]} \\ &\quad - \frac{B}{2} \exp[-(\mathrm{i}\hat{k}/\alpha_1)\ln 2] \frac{\Gamma(\mathrm{i}\hat{k}/\alpha_1)\Gamma(\frac{3}{2}-\kappa)(-1)^{-\kappa}}{\Gamma[\frac{1}{2}-\frac{1}{2}(\kappa+\lambda)+\mathrm{i}\hat{k}/2\alpha_1]\Gamma[\frac{1}{2}+\frac{1}{2}(\lambda-\kappa+1)+\mathrm{i}\hat{k}/2\alpha_1]}. \end{aligned} \tag{2.22} $$

We can easily see that (2.22) satisfy the unitarity of the scattering matrix which is a U(2) matrix

$$ |A_-|^2 + |B_+|^2 = |A_+|^2 + |B_-|^2 $$

and the quasiunitarity of the transfer matrix which is a SU(1, 1) matrix

$$ |A_+|^2 - |B_+|^2 = |A_-|^2 - |B_-|^2 $$

provided $\kappa = 0, \pm 2, \pm 4 \ldots$. This is not surprising since asymptotically the g terms in (2.3) and the m terms in (2.8) disappear and the presence of one of the two potential terms in (2.1) becomes irrelevant. We now normalise $\psi_\pm(r)$ such that $A_- = 1$ and $B_+ = 0$. This fixes A and B and consequently, after some straightforward computations, we obtain the reflection and transmission coefficients as

$$|B_-|^2=|R|^2=\frac{1}{1+x^2}\qquad |A_+|^2=|T|^2=x^2/(1+x^2)$$
$$x=\frac{\{\tan[\frac{1}{2}\pi(\lambda-\kappa+1)]+\cot[\frac{1}{2}\pi(\kappa+\lambda+1)]\}\tanh\pi\hat{k}/2\alpha_1}{1-\tan[\frac{1}{2}\pi(\lambda-\kappa+1)]\cot[\frac{1}{2}\pi(\kappa+\lambda+1)]\tanh\pi\hat{k}/2\alpha_1}. \tag{2.23}$$

For the case $\kappa=0, \pm2, \pm4, \ldots$ we obtain

$$|R|^2=\frac{\sin^2\pi\lambda}{\sin^2\pi\lambda+\sinh^2(\pi\hat{k}/\alpha_1)}$$
$$|T|^2=\frac{\sinh^2(\pi\hat{k}/\alpha_1)}{\sin^2\pi\lambda+\sinh^2(\pi\hat{k}/\alpha_1)} \tag{2.24}$$

which are in agreement with standard results (Flügge 1971).

3. Morse–Rosen equation and Eckart equation

The Morse-Rosen equation for polyatomic molecules is

$$\left[\frac{\partial}{\partial x^2}+\left(\frac{2MU_0}{\hbar^2}\right)\frac{1}{\cosh^2\alpha_1 x}-\left(\frac{2MB_0}{\hbar^2}\right)\tanh\alpha_1 x+\frac{2ME}{\hbar^2}\right]\Phi(x)=0\qquad x\in[0,\infty). \tag{3.1}$$

This equation can be algebraised by using the factorisation (Nieto 1978) either of type A or of type E. The algebraic ladder operators from the factorisation of type A raise or lower the eigenvalues (quantum numbers) of the self-adjoint operators which are the elements of a Lie algebra, as we have seen in the case of first and second Pöschl-Teller equations. But those operators from the factorisation of type E raise or lower the eigenvalues of the invariant operators (Casimir operators) in the enveloping algebra of a Lie algebra. We show these equivalent algebraisations for (3.1).

First, in (3.1) we use the following reparametrisations and substitutions:

$$2ME/\hbar^2\alpha_1^2=-p^2-q^2\qquad 2MU_0/\hbar^2\alpha_1^2=-\Delta-\tfrac{1}{4}$$
$$2MB_0/\hbar^2\alpha_1^2=-2pq$$
$$\alpha_1 x+\mathrm{i}\pi/2\leftrightarrow\ln\tanh(z/2)$$
$$\Phi(x)\leftrightarrow(\mathrm{i}/\sinh z)^{1/2}\varphi(z). \tag{3.2}$$

Consequently (3.1) becomes

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{\sinh^2 z}[(p-\tfrac{1}{2})(p+\tfrac{1}{2})+q^2+2pq\cosh z]+\Delta\right)\varphi(z)=0$$

or

$$\left[\frac{\partial^2}{\partial z^2}-\frac{1}{4}\left(\frac{(p+q+\frac{1}{2})(p+q-\frac{1}{2})}{\sinh^2 z/2}-\frac{(p-q-\frac{1}{2})(p-q+\frac{1}{2})}{\cosh^2 z/2}\right)+\Delta\right]\varphi(z)=0. \tag{3.3}$$

This equation is identical to the second Pöschl-Teller equation (3.2) with the following identifications:

$$\begin{array}{ccc} \text{Pöschl-Teller} & & \text{Morse-Rosen} \\ m & \longleftrightarrow & p \\ g & \longleftrightarrow & q \\ \Lambda & \longleftrightarrow & \Delta \\ \beta & \longleftrightarrow & z \\ \psi(\beta) & \longleftrightarrow & \varphi(z). \end{array} \tag{3.4}$$

Thus the algebraisation by factorisation of type A can be carried out as before. From (2.4) and (3.2) we obtain the energy spectrum

$$E_n = -\left[\frac{\hbar^2\alpha_1^2}{8M}(D_n^2) + \frac{2MB_0^2}{\hbar^2\alpha_1^2}\left(\frac{1}{D_n^2}\right)\right]$$
$$D_n = 2p = 2(s-n) \qquad [s-(MB_0/\hbar^2\alpha_1^2)] \geqslant n = 0, 1, 2, \ldots \tag{3.5}$$
$$s = l - 1 = -\tfrac{1}{2} + \tfrac{1}{2}[1 + 8MU_0/\hbar^2\alpha_1^2]^{1/2}.$$

Furthermore, from (3.2) and (2.14) we obtain the solution to (3.1) as

$$\Phi(x) = (2s+1)^{1/2} V_{p,q}^{s+1}(-\mathrm{i}\pi - z)$$
$$\sinh z = \mathrm{i}/\cosh \alpha_1 x \qquad \cosh z = -\tanh \alpha_1 x \tag{3.6}$$
$$\int_0^\infty \Phi^*(x)\Phi(x) \cosh \alpha_1 x \, \mathrm{d}x = 1$$

where $\cosh \alpha_1 x \, \mathrm{d}x$ is the normalised Haar measure induced by the Bargmann functions. Using (2.15) we obtain $\Phi(x)$ explicitly as

$$\Phi_n(x) = \left[\alpha_1(2s+1)2^{2n-2s}\binom{s+q}{n}\binom{2s-n}{s-q}\right]^{1/2}$$
$$\times \exp(q\alpha_1 x)(\cosh \alpha_1 x)^{-s+n}$$
$$\times {}_2F_1[-n, 2s+1-n; s-n+q+1; \mathrm{e}^{\alpha_1 x}/(\mathrm{e}^{\alpha_1 x} + \mathrm{e}^{-\alpha_1 x})]. \tag{3.7}$$

Our results (3.5) and (3.7) are in agreement with earlier calculations (Weyl 1910) (the extra term $(p^2-q^2)/p$ of Nieto's normalisation does not appear in our normalisation due to the Haar measure (3.6)). Our algebraisation using the factorisation type A is somewhat artificial (Miller 1964, 1968) (in fact, according to the Infeld-Hull classification the parameter q is artificially introduced to achieve factorisation type A). The complex substitution $\alpha_1 x + \mathrm{i}\pi/2 \leftrightarrow \ln \tanh(z/2)$ is similar to Weyl's unitarity trick.

The scattering solutions of (3.1) are obtained by taking $p \to \mathrm{i}\nu_p$ and $q \to -\mathrm{i}\nu_q$; $(\nu_p, \nu_q) \in (-\infty, \infty)$ so that the continuous energy spectrum from (3.5) is given by

$$E_{\nu_p} = \frac{\hbar^2\alpha_1^2}{2M}\nu_p^2 + \frac{MB_0^2}{2\hbar^2\alpha_1^2}\frac{1}{\nu_p^2}. \tag{3.8}$$

The algebraisation as in (2.3)-(2.11) further implies that we consider $D_l^\mp$ discrete series unitary representations of SO(2, 1) in a continuous basis, where a non-compact operator is diagonal (i.e. P_i are similar to M_i in (2.7) and Q_i are similar to G_i in (2.9))

$$P_2\psi_{\nu_p,\nu_q} = \nu_p\psi_{\nu_p,\nu_q} \qquad Q_2\psi_{\nu_p,\nu_q} = \nu_q\psi_{\nu_p,\nu_q}$$

and the corresponding matrix element we consider is the matrix element of $\exp(-izP_1)$ or $\exp(-izQ_1)$ between the respective bases. These matrix elements have already been obtained (Barut and Phillips 1968, Lindblad and Nagel 1970). We give below the solutions which are normalised to reproduce the correct asymptotic behaviour:

$$\begin{aligned}\Phi_\nu(x)=\frac{\exp[(\pi/2)(\nu_p-\nu_q)]}{2\pi}&\bigg[\frac{(-1)^{1-2s}}{\cot\pi(s-i\nu_q)}\left(\frac{\Gamma(s+i\nu_p+1)\Gamma(s-i\nu_q+1)}{\Gamma(s-i\nu_p+1)\Gamma(s+i\nu_q+1)}\right)^{1/2}\\ &\times(-1)^{(i/2)(\nu_p-\nu_q)}\exp(-i\alpha_1x\nu_q)(2\cosh\alpha_1x)^{-i\nu_p}\Gamma(i\nu_q-i\nu_p)\\ &\times{}_2F_1[-s+i\nu_p,s+i\nu_p+1;i\nu_p-i\nu_q+1;e^{\alpha_1x}/(e^{\alpha_1x}+e^{-\alpha_1x})]\\ &+\frac{(-1)^{2s}}{\cot\pi(s-i\nu_p)}\left(\frac{\Gamma(s+i\nu_q+1)\Gamma(s-i\nu_p+1)}{\Gamma(s-i\nu_q+1)\Gamma(s+i\nu_p+1)}\right)^{1/2}\\ &\times(-1)^{(i/2)(\nu_q-\nu_p)}\exp(-i\alpha_1x\nu_p)(2\cosh\alpha_1x)^{-i\nu_q}\Gamma(i\nu_p-i\nu_q)\\ &\times{}_2F_1[-s+i\nu_q,s+i\nu_q+1;i\nu_q-i\nu_p+1;e^{\alpha_1x}/(e^{\alpha_1x}+e^{-\alpha_1x})]\bigg]\end{aligned}\tag{3.9}$$

where we have made use of the representation function obtained by Rühl (1970).

Before we algebraise the Morse–Rosen equation using the factorisation of type E, we discuss the closely related Eckart equation (Eckart 1930) which was used to describe the penetration of a potential barrier by electrons. The Eckart equation is

$$\left(\frac{\partial^2}{\partial x^2}+\frac{2MA}{\hbar^2}\frac{\xi}{1-\xi}+\frac{2MB}{\hbar^2}\frac{\xi}{(1-\xi)^2}+\frac{2MW}{\hbar^2}\right)f(x)=0$$

$$\xi=-\exp(2\pi x/l)\qquad A,B,l\text{ constants}, B\geqslant 0.\tag{3.10}$$

We can rewrite (3.10) as

$$\left(\frac{\partial^2}{\partial x^2}-\frac{MB}{2\hbar^2}\frac{1}{\cosh^2(\pi x/l)}-\frac{MA}{\hbar^2}\tanh(\pi x/l)+\frac{M}{\hbar^2}(2W-A)\right)f(x)=0\tag{3.11}$$

which is similar to (3.1). This means that in the Eckart equation (3.11) we can use similar substitutions and parametrisations as in (3.2):

$$\begin{aligned}&(2MW/\hbar^2-MA/\hbar^2)l^2/\pi^2=-a^2-b^2\\ &(MB/2\hbar^2)(l^2/\pi^2)=\Delta^1+\tfrac14\qquad (MA/2\hbar^2)(l^2/\pi^2)=-2ab\\ &\pi x/l+i\pi/2\leftrightarrow\ln\tanh(z/2)\\ &f(x)\leftrightarrow(i/\sinh z)^{1/2}g(z).\end{aligned}\tag{3.12}$$

Consequently we obtain an equation similar to (3.3)

$$\left[\frac{\partial^2}{\partial z^2}-\frac14\left(\frac{(a+b+\frac12)(a+b-\frac12)}{\sinh^2(z/2)}-\frac{(a-b-\frac12)(a-b+\frac12)}{\cosh^2(z/2)}\right)+\Delta^1\right]g(z)=0.\tag{3.13}$$

This means that we can perform an algebraisation exactly identical to the one we developed for the second Pöschl–Teller equation and from (2.4) we obtain the important condition

$$(MB/2\hbar^2)(l^2/\pi^2)-\tfrac14=\Delta^1=-(l-\tfrac12)^2.\tag{3.14}$$

In order to maximise the potential barrier Eckart assumes (see figure 1 of Eckart 1930) $B\geqslant 0\Rightarrow\Delta^1\geqslant-\frac14$. For $\Delta^1=-\frac14\Rightarrow l=0$, there may exist a few bound states given by $D_l^\mp$ discrete series representations of SO(2, 1). However, the existence of such bound states may be removed by taking B sufficiently large. Thus we are led to the continuous principal series representations C_l^δ of SO(2, 1): $l=\frac{1}{2}+\mathrm{i}t$, $t\in(0,\infty)$; for $\delta=0$, $(a,b)=0,\pm1,\pm2,\ldots$ or for $\delta=\frac{1}{2}$, $(a,b)=\pm\frac{1}{2},\pm\frac{3}{2},\ldots$. Now, from the energy relation given (3.12) we see that there exist two possibilities. First, for $W<A/2$ there exist asymptotically plane wave solutions (pulses) with fixed discrete energy. However, the second possibility, for which $W>A/2$, is very interesting. In this case in (3.12) we use the analytic continuation $a\to\mathrm{i}\nu_a$, $b\to-\mathrm{i}\nu_b$, $(\nu_a,\nu_b)\in(-\infty,\infty)$. The continuous energy spectrum is given by

$$W=\frac{\hbar^2\pi^2}{2Ml^2}\nu_a^2+\frac{MA^2l^2}{8\hbar^2\pi^2}\frac{1}{\nu_a^2}+\frac{A}{2}. \tag{3.15}$$

Thus we consider the continuous principal series representations C_l^δ on a continuous basis (A_i are similar to M_i in (2.7) and B_i are similar to G_i in (2.9))

$$A_2f_{\nu_a,\nu_b}(z)=\nu_af_{\nu_a,\nu_b}(z) \qquad B_2f_{\nu_a,\nu_b}(z)=\nu_bf_{\nu_a,\nu_b}(z)$$

where $f_{\nu_a,\nu_b}(z)$ are notationally similar to $\psi_{m,g}(\beta)$ in (2.11). The spectrum of A_2, B_2 is the real line with multiplicity two—there exist two eigenstates for each eigenvalue. This is because there exists an outer automorphism (parity) of the Lie algebra $\{A_i\}\to\{\tilde{A}_i\}$; $\{B_i\}\to\{\tilde{B}_i\}$ where

$$\{\tilde{A}_1,\tilde{A}_2,\tilde{A}_3\}=-\{-A_1,A_2,-A_3\}$$
$$\{\tilde{B}_1,\tilde{B}_2,\tilde{B}_3\}=\{-B_1,B_2,-B_3\}.$$

In the case of the continuous principal series C_l^δ (and also for the supplementary series) this automorphism can be realised explicitly by

$$\tilde{A}_i=PA_iP^{-1} \qquad Pf_{\nu_a,\nu_b}=\exp(\mathrm{i}\pi\nu_a)f_{-\nu_a,\nu_b}$$
$$\tilde{B}_i=PB_iP^{-1} \qquad Pf_{\nu_a,\nu_b}=\exp(\mathrm{i}\pi\nu_b)f_{\nu_a,-\nu_b}.$$

Therefore $P^2=1$ and the eigenvalues of P are ±1 which we denote as ε_a, ε_b. As $[P,A_2]=[P,B_2]=0$, (P and A_2) and (P and B_2) can be diagonalised simultaneously.

The matrix elements we consider in this case are those of $\exp(-\mathrm{i}A_1z)$ or $\exp(-\mathrm{i}B_1z)$ between the two continuous bases of eigenvalues ν_a and ν_b with respective multiplicity ε_a and ε_b. These representation functions are already known (Barut and Phillips 1968). Thus from (3.6), (3.9) and (3.12) we obtain the solutions which are as usual normalised to reproduce the correct asymptotic behaviour

$$\begin{aligned}f_\nu(x)=&(1/4\pi^2)[\cosh\pi(\nu_b+t)+\varepsilon_a\varepsilon_b\cosh\pi(\nu_a+t)+\mathrm{i}\varepsilon_b\sinh\pi(\nu_b-\nu_a)]\\&\times\Gamma(\tfrac{1}{2}+\mathrm{i}t+\mathrm{i}\nu_a)\Gamma(\tfrac{1}{2}-\mathrm{i}t-\mathrm{i}\nu_b)(-1)^{\mathrm{i}(\nu_a-\nu_b)/2}\exp(-\mathrm{i}\pi x\nu_b/l)\\&\times[2\cosh(\pi x/l)]^{-\mathrm{i}\nu_a}\Gamma(\mathrm{i}\nu_b-\mathrm{i}\nu_a)\\&\times{}_2F_1[\tfrac{1}{2}-\mathrm{i}t+\mathrm{i}\nu_a,\tfrac{1}{2}+\mathrm{i}t+\mathrm{i}\nu_a;\mathrm{i}\nu_a-\mathrm{i}\nu_b+1;\mathrm{e}^{\pi x/l}/(\mathrm{e}^{\pi x/l}+\mathrm{e}^{-\pi x/l})]\\&+(1/4\pi^2)[\cosh\pi(\nu_b-t)+\varepsilon_a\varepsilon_b\cosh\pi(\nu_a-t)\\&+(-1)^{2\delta}\mathrm{i}\varepsilon_b\sinh\pi(\nu_a-\nu_b)]\\&\times(-1^{\mathrm{i}(\nu_b-\nu_a)/2}\exp(-\mathrm{i}\pi x\nu_a/l)(2\cosh\pi x/l)^{-\mathrm{i}\nu_b}\Gamma(\mathrm{i}\nu_a-\mathrm{i}\nu_b)\\&\times{}_2F_1[\tfrac{1}{2}-\mathrm{i}t+\mathrm{i}\nu_b,\tfrac{1}{2}+\mathrm{i}t+\mathrm{i}\nu_b;\mathrm{i}\nu_b-\mathrm{i}\nu_a+1;\mathrm{e}^{\pi x/l}/(\mathrm{e}^{\pi x/l}+\mathrm{e}^{-\pi x/l})]\end{aligned} \tag{3.16}$$

where $\delta=0,\frac{1}{2}$ depending on the C_l^δ representation and we have made use of the representation function obtained by Mukunda (1969). Our result (3.16) up to a phase term is in agreement with Eckart's result.

Thus we have algebraised all three equations in a unified manner using factorisation of type A. We have summarised in table 1 the representation of SO(1, 2) used in each case.

Table 1. Representations of SU(2, 1) occurring in various equations.

	Bound states		Scattering states	
Equation	Energy label	Representations	Energy label	Representations
Pöschl-Teller equation (2.1)	Casimir label l	$D_l^{\mp}$ on discrete basis	Casimir label $l=\frac{1}{2}+\mathrm{i}t$	C_l^{δ} on discrete basis $\delta=0,\frac{1}{2}$
Morse-Rosen equation (3.1)	Function eigenvalues of compact operator	$D_l^{\mp}$ on discrete basis	Eigenvalues of non-compact operator	$D_l^{\mp}$ on continuous basis
Eckart equation (3.10) $B\geqslant 0$, $W\geqslant A/2$	No discrete spectrum		Eigenvalues of non-compact operator	C_l^{δ} on continuous basis $l=\frac{1}{2}+\mathrm{i}t$; $\delta=0,\frac{1}{2}$

We now algebraise (3.1) using factorisation of type E. While factorisations of type A give rise to ladder operators which raise or lower the eigenvalues (for example p, q) of the operators within the Lie algebra, the factorisation of type E brings about ladder operators which raise or lower the Casimir label (for example l) of the same Lie algebra and thus the latter ladder operators are in the enveloping algebra. The algebraisation of (3.2) by factorisation of type A has led us to two SO(2, 1) groups. We distinguish them as $\mathrm{SO}(2,1)_p$ and $\mathrm{SO}(2,1)_q$ corresponding to the p and q eigenvalues. Since, as we have seen in (2.10), the generators of $\mathrm{SO}(2,1)_p$ commute with those of $\mathrm{SO}(2,1)_q$, we have the complete group structure as $\mathrm{SO}(2,1)_p\otimes\mathrm{SO}(2,1)_q\approx\mathrm{SO}(2,2)$. We know (Kihlberg 1965) that SO(2, 2) covers $\mathrm{SO}(2,1)_p\otimes\mathrm{SO}(2,1)_q$ twice since the former contains a centre of order two while the latter has no centre. Since the Casimir operators Q_p and Q_q of $\mathrm{SO}(2,1)_p$ and $\mathrm{SO}(2,1)_q$ respectively are equal, under the isomorphism given above, the second quadratic Casimir product (Q'') of SO(2, 2) vanishes while the first quadratic Casimir product Q' is proportional to Q_p and Q_q. The invariant product Q' may be constructed (similar to Q_p and Q_q) in terms of bilinear forms of the l raising and lowering ladder operators provided these operators form an algebra. We will see below that they do not close under a commutation relation and we will see that these bilinear products are elements in the enveloping algebra of $\mathrm{SO}(2,1)\otimes\mathrm{SO}(2,1)\sim\mathrm{SO}(2,2)$.

We use the parametrisations (3.2) in (3.1) and from factorisation of type W we obtain the ladder operators

$$\begin{aligned}
L^{+}\Phi_l(x) &= \left(-\frac{\partial}{\partial x}+l\alpha_1\tanh\alpha_1 x-\frac{pq\alpha_1}{l}\right)\Phi_l(x)\\
&= \frac{\alpha_1}{l}[(l-p)(l+p)(l-q)(l+q)]^{1/2}\Phi_{l+1}(x)\\
L^{-}\Phi_l(x) &= \left(\frac{\partial}{\partial x}+(l-1)\alpha_1\tanh\alpha_1 x-\frac{pq\alpha_1}{(l-1)}\right)\Phi_l(x)\\
&= \frac{\alpha_1}{(l-1)}[(l-1-p)(l-1+p)(l-1-q)(l-1+q)]^{1/2}\Phi_{l-1}(x)
\end{aligned}\tag{3.17}$$

$$[L^+, L^-] = \alpha_1^2\left[\left((l-1)^2 + \frac{p^2q^2}{(l-1)^2}\right) - \left(l^2 + \frac{p^2q^2}{l^2}\right)\right].$$

We see that $L^{\pm}$ do not close under an algebra. In order to see the implication of the SO(2, 1)⊗SO(2, 1) group structure, we use the substitution (3.2) in z variables and obtain

$$\begin{aligned} L^+\Phi_l(x) &= \left(\sinh z\frac{\partial}{\partial z} + l\cosh z + \frac{pq}{l}\right)\Phi_l(x) \\ &= -\frac{1}{l}[(l-p)(l+p)(l-q)(l+q)]^{1/2}\Phi_{l+1}(x) \\ L^-\Phi_l(x) &= \left(\sinh z\frac{\partial}{\partial z} - (l-1)\cosh z - \frac{pq}{(l-1)}\right)\Phi_l(x) \\ &= \frac{1}{(l-1)}[(l-1-p)(l-1+p)(l-1-q)(l-1+q)]^{1/2}\Phi_{l-1}(x). \end{aligned} \tag{3.18}$$

The recurrence relations for Bargmann functions from SO(2, 1)⊗SO(2, 1) are given by (Schneider and Wilson 1979, Barut and Wilson 1976, Basu and Wolf 1983)

$$\begin{aligned} &\left(\sinh\theta\frac{\partial}{\partial\theta} + l\cosh\theta - \frac{n'n}{l}\right)V^l_{n'n}(\theta) = -\frac{1}{l}[(l-n')(l+n')(l-n)(l+n)]^{1/2}V^{l+1}_{n'n}(\theta) \\ &\left(\sinh\theta\frac{\partial}{\partial\theta} - (l-1)\cosh\theta + \frac{n'n}{(l-1)}\right)V^l_{n'n}(\theta) \\ &\qquad = \frac{1}{(l-1)}[(l-1-n')(l-1+n')(l-1-n)(l-1+n)]^{1/2}V^{l-1}_{n'n}(\theta). \end{aligned} \tag{3.19}$$

On comparison we find that $\Phi_l(x)$ is given by the Bargmann function $V^l_{p,-q}(z)$ as obtained earlier in (3.6). The energy spectrum is certainly given by the parametrisations (3.2) and furthermore the algebraisation using factorisation of type E for the continuum part of (3.1) and for the Eckart equation follows immediately as (3.1) and (3.11) are identical except for the coefficients.

4. Discussion

As in I, we found ((2.3) and (2.8)) that the SU(1, 1)⊗SU(1, 1) algebras describe the fixed energy states of a family of systems with quantised coupling constants κ and λ, as some kind of periodic table of elements. The energy range is finite, determined by the range $l_{\min}$ to $l_{\max}$, or $n = 0$ to $n_{\max} = (\lambda - \kappa)/2$ (see (2.4)). In addition we can change energy, or l, for fixed coupling constants κ and λ as the type E factorisation shows ((3.17) and (3.18)). Again we see that our family of systems can be embedded in an SO(4, 2) (Barut *et al* 1987b). For the first Pöschl-Teller equation the range of discrete energy is infinite but we have a finite family of systems, while for the second Pöschl-Teller equation we have an infinity of systems, but a finite number of discrete energy levels. In other words, the role of the subgroups is interchanged, SU(1, 1)⊗SU(1, 1) against SU(2)⊗SU(2). This accounts for the analytic continuation between the two cases.

References

Bargmann V 1947 *Ann. Math.* **48** 568
Barut A O and Fronsdal C 1965 *Proc. R. Soc.* A **287** 532
Barut A O, Inomata A and Wilson R 1987a *J. Phys. A: Math. Gen.* **20** 4075-82
—— 1987b *J. Math. Phys.* **28** 605
Barut A O and Phillips E C 1968 *Commun. Math. Phys.* **8** 52
Barut A O and Wilson R 1976 *J. Math. Phys.* **17** 900
Basu D and Wolf K B 1983 *J. Math. Phys.* **24** 478
Biedenharn L C and Louck J D 1981 *Angular Momentum in Quantum Physics* (Reading, MA: Addison-Wesley)
Eckart C 1930 *Phys. Rev.* **35** 1303
Flügge S 1971 *Practical Quantum Mechanics I* (Berlin: Springer) p 95
Goldberg J N, MacFarlane A J, Newman E T, Rohrlich F and Sudarshan E V G 1967 *J. Math. Phys.* **8** 2155
Kihlberg A 1965 *Ark. Fys.* **30** 121
Kunze R A and Stein E M 1960 *Am. J. Math.* **82** 1
Lindblad G and Nagel B 1970 *Ann. Inst. H Poincaré* A **13** 27
Miller W Jr 1964 *Mem. Am. Math. Soc.* no 50
—— 1968 *Lie Theory and Special Functions* (New York: Academic)
Mukunda N 1969 *J. Math. Phys.* **10** 2086, 2092
Newman E T and Penrose R 1966 *J. Math. Phys.* **7** 863
Nieto M M 1978 *Phys. Rev.* A **17** 1273
Rühl W 1970 *The Lorentz Group and Harmonic Analysis* (New York: Benjamin)
Schneider C K E and Wilson R 1979 *J. Math. Phys.* **20** 2380
Toller M 1965 *Nuovo Cimento* **37** 631
Weyl H 1909 *Königlichen Gesellschaft der Wissenschaften zu Gottingen, Nachrichten* 37
—— 1910 *Königlichen Gesellschaft der Wissenschaften zu Göttingen, Nachrichten* 442